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LETTER TO THE EDITOR

On completely positive nonlinear dynamical semigroups

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Abstract. The general form of infinitesimal generators of completely positive nonlinear semigroups is discussed. It is argued that the condition of complete positiveness is too restrictive.

Dynamical processes in statistical mechanics are often described by nonlinear kinetic equations. As examples one can consider Hartree-type equations or quantum Boltzmann-type equations. It is natural to assume that the solutions of nonlinear evolution equations have a common feature, which is the completely positive semigroup property. In this letter I shall give a general characterisation of infinitesimal generators of the nonlinear dynamical semigroup; then I shall discuss a Hartree-type equation. This example will show that the condition of complete positiveness is a very restrictive one.

In order to present our results it is convenient to introduce the following notation. For a completely positive map T from a unitary C^*-algebra $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$ we shall define T such that for every $n>0$, the n-square $\mathscr{B}$-valued matrix $[T(a_{ij})]_{i,j=1}^{n}$ is positive whenever an n-square $\mathscr{A}$-valued matrix $[a_{ij}]$ is positive. We define as a nonlinear dynamical semigroup a one-parameter family of maps $T:\mathscr{A}\rightarrow\mathscr{A}$ which fulfil the following conditions:

$$T_t \text{ is a completely positive contractive map} \tag{1}$$

$$T_t(1)=1 \tag{2}$$

$$T_t\circ T_s = T_{t+s} \tag{3}$$

$$\text{for every } a\in\mathscr{A} \text{ the function } t\rightarrow T_t(a) \text{ is uniformly continuous in } a \tag{4}$$

where $t, s\geqslant 0$. In other words $\{T_t\}_{t\geqslant 0}$ is a nonlinear analogue of the so called Lindblad-type dynamical semigroup.

For each positive integer n let $\mathscr{A}^{\otimes^n}$ be the n-fold tensor product of $\mathscr{A}$. The C^*-subalgebra of $\mathscr{A}^{\otimes^n}$ generated by elementary tensors of the form

$$a^{(n)} = a\otimes a\otimes\ldots\otimes a$$

will be denoted by $\mathscr{A}^n$. We define (see Arveson [1] for details) $e^{\mathscr{A}}\otimes e^{\bar{\mathscr{A}}}$ to be the tensor product of direct sums of C^*-algebras

$$e^{\mathscr{A}} = \sum_{n=0}^{\infty}\mathscr{A}^n \qquad \text{and} \qquad e^{\bar{\mathscr{A}}} = \sum_{n=0}^{\infty}\bar{\mathscr{A}}^n$$

where $\mathscr{A}^0$ is defined as $\mathbb{C}$, and $\bar{\mathscr{A}}$ is a conjugate C^*-algebra with respect to $\mathscr{A}$. $\Gamma(\bar{\Gamma})$ denotes a map of the closed unit ball of $\mathscr{A}$ $(\bar{\mathscr{A}})$ into $e^{\mathscr{A}}$ $(e^{\bar{\mathscr{A}}})$ defined by

$$\Gamma(a)=(1, a, a\otimes a, \ldots)$$
$$\bar{\Gamma}(\bar{a})=(1, \bar{a}, \bar{a}\otimes\bar{a}, \ldots).$$

Remark. All C^*-algebraic tensor products are taken with respect to the largest C^* cross-norm.

Now, let us consider a completely positive map T: unit ball of $\mathscr{A}\to\mathscr{A}$. The main result of [2] can be formulated, in a slightly modified way, as follows.

Theorem. Let T: ball $\mathscr{A}\to\mathscr{A}$ be a bounded completely positive map. There exists a positive linear map τ of the C^*-algebra $e^{\mathscr{A}}\otimes e^{\bar{\mathscr{A}}}$ into $\mathscr{A}$ such that

$$T(a)=\tau(\Gamma(a)\otimes\bar{\Gamma}(\bar{a})). \tag{5}$$

Corollary. Let $\{T_t\}$ be a nonlinear dynamical semigroup on $\mathscr{A}$. If $\{T_t\}$ has a well defined infinitesimal generator $\mathscr{L}$ then

$$\mathscr{L}(a)=L(\Gamma(a)\otimes\bar{\Gamma}(\bar{a})) \qquad a\in D(\mathscr{L})\cap \text{ball } \mathscr{A} \tag{6}$$

for some linear map L of $e^{\mathscr{A}}\otimes e^{\bar{\mathscr{A}}}$ into $\mathscr{A}$.

Proof. It is enough to differentiate $T_t(a)=\tau_t(\Gamma(a)\otimes\bar{\Gamma}(\bar{a}))$.

Remarks. (i) In general a generator of a semigroup of nonlinear contractions does not have to be defined as a densely defined map. Moreover, there are semigroups of contractions on some Banach spaces which have no generators in any sense [3]. Therefore our assumption about differentiability of T_t is the essential one.

(ii) If one omits the contribution of $\bar{\mathscr{A}}$ to the general form of a completely positive nonlinear map T_t (cf theorem, see also [1]) then its remaining part imitates the second quantisation map. Let us assume, for a moment, that only such maps have a physical sense. It is easy to observe that for such maps the set of linear terms has the semigroup property. If one assumes non-triviality of linear terms, then the general form of generators of such nonlinear dynamical semigroups is given as

$$\mathscr{L}(a)=L(a)+M(a) \tag{7}$$

where $L(\cdot)$ is a infinitesimal generator of a linear semigroup and $M(\cdot)$ is a (nonlinear) completely positive map. However, solvable models suggest that this is not the case [4, 5]. Therefore we conclude that, in general, anti-multilinear terms of the general form of T_t are essential ones.

(iii) Examples of infinitesimal generators of the form given in (6) are provided by Boltzmann-type equations and some Hartree-type equations [4, 5].

Among nonlinear equations of the Hartree-type the following one seems to be very interesting (cf [6])

$$\frac{\mathrm{d}}{\mathrm{d}t}a_t=\mathrm{i}|\mathrm{Tr}\, a_tQ|^{\alpha}[Q, a_t] \tag{8}$$

where $a\in B(H)$, H is a finite dimensional Hilbert space, $Q\in B(H)$ and $Q=Q^*$, $[\cdot,\cdot]$ stands for the commutator, $\alpha\in R$. Such an equation can be obtained as a slight generalisation of equations considered in the standard model for a mean-field limit in a lattice spin system.

Clearly, the right-hand side of (8) is not of the form described by (6). On the other hand let us observe that $\mathrm{Tr}(a_t Q)$ does not depend upon t. Hence the solution of (8) can be written as

$$V_t(a) = a_t = \exp[\mathrm{i}(\mathrm{Tr}\, aQ)^\alpha Qt]a \exp[-\mathrm{i}(\mathrm{Tr}\, aQ)^\alpha Qt]. \tag{9}$$

Let us examine, for some fixed t_0, the complete positivity of V_{t_0}. If V_{t_0} is completely positive map then it should be of the form (cf our earlier theorem; see also [2])

$$V_{t_0}(a) = \sum_{m,n\geq 0} V_{t_0}^{mn}(a) \tag{10}$$

where $V_{t_0}^{mn}$ are completely positive maps and $V_{t_0}^{mn}(\lambda a) = \lambda^m \bar{\lambda}^n V_{t_0}^{mn}(a)$. Moreover, the decomposition (10) is the unique one. Hence [2]

$$V_{t_0}^{mn}(a) = \frac{1}{m!\,n!} \frac{\partial^{m+n}}{\partial \lambda^m\, \partial \bar{\lambda}^n} V_{t_0}(\lambda a)|_{\lambda=0}. \tag{11}$$

By definition of $V_{t_0}(\lambda a)$ we have

$$V_{t_0}(\lambda a) = \sum_{k,l\geq 0} \frac{\lambda^{(\alpha/2)(k+l)+1} \bar{\lambda}^{(\alpha/2)(k+l)}}{k!\,l!} |\mathrm{Tr}\, aQ|^{\alpha(k+l)} (\mathrm{i}Qt_0)^k a(-\mathrm{i}Qt_0)^l.$$

Thus, in the expression (10) only those $V_{t_0}^{mn}(a)$ can survive, for which $n+1=m$. Furthermore

$$V_{t_0}^{m+1\,n}(a) = \sum_{k+1=2m/\alpha} [(\alpha/2)(k+l)+1][(\alpha/2)(k+l)]^2 \ldots [(\alpha/2)(k+l)-m+1]^2$$
$$\times \frac{|\mathrm{Tr}\, aQ|^{\alpha(k+l)}}{k!\,l!} (\mathrm{i}Qt_0)^k a(-\mathrm{i}Qt_0)^l.$$

In particular, for $\alpha = 2$

$$V_{t_0}^{21}(a) = -\mathrm{i}|\mathrm{Tr}\, aQ|^2 t_0[a, Q]. \tag{12}$$

Clearly, $V_{t_0}^{21}$ is not a completely positive map. Therefore one can infer that our assumption about V_{t_0} is false. Consequently, there are solutions of the considered Hartree-type equations which are not completely positive. So we can conclude that the assumption of complete positiveness for dynamical semigroup is very restrictive but if this condition is met, then the infinitesimal generator of semigroup has a relatively simple form.

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